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Percolation in the random cluster process and Q-state Potts model

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Abstract. The critical probabilities $p_T = p_T(Q)$ and $p_H = p_H(Q)$ of the random cluster model with parameter $Q \geqslant 1$ on the square lattice are shown to satisfy $p_T \leqslant \sqrt{Q}/(1+\sqrt{Q}) \leqslant p_H$, $p_H \leqslant Q(1-p_T)/(p_T + Q(1-p_T))$. The last inequality is the Q-analogue of the relation $p_T + p_H \leqslant 1$ of ordinary ($Q = 1$) percolation, though of course in this case equality holds.

1. Introduction

The main purpose of this paper is to examine the percolatory behaviour of the random cluster process on the square lattice. This process, which was introduced by Fortuin and Kasteleyn (1972) contains ordinary percolation, the Ising model, and the Q-state Potts model as special cases, corresponding to giving Q positive integer values and complete knowledge of its critical behaviour would answer several longstanding problems of statistical physics. Here we allow Q to take arbitrary non-negative values. Thus when Q is not an integer the random cluster model is a genuine extension of the Potts model. For a discussion of the applications and physical background we refer to the papers of Aizenman *et al* (1988), Aizenman and Grimmett (1991), Bezuidenhout *et al* (1992), Fortuin (1972), Edwards and Sokal (1988), Gandolfi *et al* (1992), Sokal (1989) and Wu (1982).

This paper makes no attempt at generality but concentrates on the combinatorial aspect of the finite model and the special (and presumably easiest) case of the two-dimensional square lattice. Edge interactions (or probabilities) are assumed to be a constant p and so the fundamental quantity of interest is the percolation probability $\theta(p, Q)$ which, when $Q = 1$ reduces to the usual percolation probability, when $Q = 2$ (the Ising model case) it reduces to a version of correlated percolation studied by Hammersley and Mazzarino (1983) which was one of the original motivations for this paper.

The terminology of percolation theory and graph theory is standard and is based principally on that used in Grimmett (1989). Other terms are defined as they are encountered.

2. The random cluster model

The general random cluster model on a finite graph G (represented as the pair (V, E)) is a generalized bond percolation model on the edge set E of G defined by the probability distribution

$$\mu(A) = Z^{-1}\left(\prod_{e\in A} p_e\right)\left(\prod_{e\notin A}(1-p_e)\right)Q^{k(A)} \qquad (A \subseteq E) \tag{2.1}$$

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where $k(A)$ is the number of connected components (including isolated vertices) of the subgraph $G : A = (V, A)$, p_e $(0 \leqslant p_e \leqslant 1)$ are parameters associated with each edge of G, $Q \geqslant 0$ is a parameter of the model, and Z is the normalizing constant introduced so that

$$\sum_{A \subseteq E} \mu(A) = 1.$$

We will sometimes use $\omega(G)$ to denote the random configuration produced by μ, and P_μ to denote the associated probability distribution.

Thus, in particular, $\mu(A) = P_\mu\{\omega(G) = A\}$. When $Q = 1$, μ is what Fortuin and Kasteleyn call a *percolation model* and when each of the p_e are made equal, say to p, then $\mu(A)$ is clearly seen to be the probability that the set of *open* edges is A in bond percolation. When $Q = 2$ we have the *Ising model* with zero magnetic field and for general positive integer Q we have the *Q-state Potts model.* Thus, in a sense, the random cluster model defines an analytic continuation of the Potts model to non-integer Q.

For an account of the many different interpretations of the random cluster model we refer to the original paper of Fortuin and Kasteleyn (1972) or Sokal (1989).

In this paper we shall be concentrating on the percolation problem when each of the p_e are equal, to say p, and henceforth this will be assumed.

Thus we will be concerned with a two parameter family of probability measures

$$\mu = \mu(p, Q) \qquad 0 \leqslant p \leqslant 1 \qquad Q > 0$$

defined on the edge set of the finite graph $G = (V, E)$ by

$$\mu(A) = p^{|A|} q^{|E \setminus A|} Q^{k(A)} / Z$$

where Z is the appropriate normalizing constant, and $q = 1 - p$.

Probably the principal reason for studying percolation in the random cluster model is its relation to phase transitions via the two-point correlation function. This was first pointed out by Fortuin and Kasteleyn (1972) and given further prominence recently by Edwards and Sokal (1988) in connection with the Swendsen–Wang algorithm (Swendsen and Wang 1987) for the Potts model. We briefly describe the connection.

Let Q be a positive integer and consider the Q-state Potts model on a finite graph G. If $\sigma = (\sigma(1), \ldots, \sigma(m))$ denotes a set of spins on the vertex set $\{1, \ldots, m\}$ of G, each spin σ_i can take a value in the set $\{1, 2, \ldots, Q\}$. The Hamiltonian $H(\sigma)$ is defined by

$$H(\sigma) = \sum_{ij} J_{ij}(1 - \delta(\sigma(i), \sigma(j)))$$

where J_{ij} are the interaction energies and the partition function is

$$Z = \sum_{\{\sigma\}} \exp[-H(\sigma)].$$

The probability of finding the system in the state σ is given by

$$P(\sigma) = \mathrm{e}^{-H(\sigma)} / Z.$$

Now let $\langle\ \rangle$ denote expectation with respect to this Gibbs distribution and the key result is that, for any pair of sites (vertices) i, j

$$\langle \delta(\sigma(i), \sigma(j)) \rangle = \frac{1}{Q} + \frac{(Q-1)}{Q} P_\mu\{i \rightsquigarrow j\} \tag{2.2}$$

where δ is the normal delta function and P_μ is the random cluster measure on G given by taking $p_e = 1 - \exp(-J_{ij})$ for each edge $e \doteq (ij)$.

The attractive interpretation of this is that the expression on the right-hand side can be regarded as being made up of two components.

The first term, $1/Q$, is just the probability that under a purely random Q-colouring of the vertices of G, i and j are the same colour. The second term measures the probability of long range interaction. Since the left-hand side of (2.2) is just the probability that i and j have the same spin (or colour) we interpret (2.2) as expressing an equivalence, when i and j are far apart, between long-range spin correlations and long-range percolatory behaviour.

Phase transition (in an infinite system) occurs at the onset of an infinite cluster in the random cluster model and corresponds to the spins on the vertices of the Potts model having a long range two-point correlation.

3. The partition function

In order to be able to calculate, or even simulate, the Gibbs state probabilities it seems to be necessary to know (or be able to approximate) the partition function Z. In the case of ordinary percolation, $Q = 1$ and $Z = 1$, but in general determining Z is demonstrably difficult as we now show.

We need first to define the *rank function* on the edge set E of a graph G. For $A \subseteq E$, the *rank* of A, $r(A)$ is defined by

$$r(A) = |V(G)| - k(A) \tag{3.1}$$

where $k(A)$ is the number of connected components of the graph $G : A$.

The key properties of the rank function that we need to note are that it is non-decreasing, integer valued and submodular, in that for $A, B \subseteq E$

$$r(A) + r(B) \geqslant r(A \cup B) + r(A \cap B). \tag{3.2}$$

The *Tutte polynomial* $T(G; x, y)$ of G is the two-variable polynomial defined by

$$T(G; x, y) = \sum_{A \subseteq E(G)} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{3.3}$$

This is not the way it was originally defined, nor is it the most convenient to use for calculations, however it will suffice at this stage. Further details and a list of properties of T including its relation to knots, percolation, codes and many other invariants may be found in Jaeger *et al* (1990).

Fortuin and Kasteleyn (1972) were aware of a relationship between the random cluster model and the Tutte polynomial which we express in the following proposition.

Proposition 3.4. For any finite graph G and subset A of $E(G)$, the Gibbs probability μ is given by

$$\mu(A) = \frac{(p/q)^{|A|} Q^{-r(A)}}{(p/Qq)^{r(E)} T(G; 1 + Qq/p, 1/q)}$$

where T is the Tutte polynomial of G, and where $q = 1 - p$.

A first consequence of this is that this shows that in a very formal sense determining the Gibbs probability μ is an intractable problem for most Q and most graphs.

This is because it was shown in Vertigan and Welsh (1992) that:

(3.5) Computing the Tutte polynomial of a planar bipartite graph is #P-hard except when $Q = 1$ or $Q = 2$.

Now to describe a problem as #P-hard means that it is at least as hard as any 'sensible' counting problem, such as determining the number of satisfying truth assignments of a Boolean formula or the number of Hamiltonian paths of a graph. In other words:

(3.6) Determining $Z(p, Q)$ and hence μ, even for bipartite planar graphs, is #P-hard except when $Q = 1$ or 2.

By Kasteleyn's algorithm (Kasteleyn 1963), we know there is a polynomial time algorithm for determining $Z(p, 2)$ for planar graphs.

An obvious quantity of interest in the random cluster model is the probability that a particular set is open. We call this the *distribution function*, denote it by λ and note:

(3.7) For fixed p, the distribution function λ is a monotone non-increasing function of Q, for $Q \geqslant 1$.

Problem. How does λ vary with Q when $0 < Q < 1$?

I do not see how to answer this but there is some slight evidence below that monotonicity extends to this region also.

The probability that w, the random subgraph of G determined by the open edges under a random cluster measure $\mu \equiv \mu(p, Q)$, is connected is known as the *reliability probability*, and is denoted by

$$\mathrm{Rel}(G) = \mathrm{Rel}(G; p, Q) = \sum_{A \in \mathrm{Sc}(G)} \mu(A)$$

where Sc denotes all subsets of edges which are spanning and connected. In other words $A \in \mathrm{Sc}(G)$ iff $r(A) = r(E)$. Using this and proposition 3.4 we get

$$\mathrm{Rel}(G) = \sum_{A \in Sc(G)} \left(\frac{p}{q}\right)^{|A|} Q^{-r(E)} \Big/ \left(\frac{Qq}{p}\right)^{-r(E)} T(G) \tag{3.8}$$

$$= \left(\frac{q}{p}\right)^{r(E)} \left(\sum_{A: r(A) = r(E)} \left(\frac{p}{q}\right)^{|A|} \right) \Big/ T(G). \tag{3.9}$$

But recall that

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E) - r(A)} (y-1)^{|A| - r(A)}$$

so that

$$\sum_{A: r(A) = r(E)} (y-1)^{|A| - r(E)} = T(G; 1, y).$$

Using this in (3.8) with $y - 1 = p/q$, gives for any finite G and any $Q \geqslant 0$, $0 \leqslant p \leqslant 1$

$$\mathrm{Rel}(G; p, Q) = \frac{T(G; 1, 1/q)}{T(G; 1 + Qq/p, 1/q)}. \tag{3.10}$$

This shows that reliability is monotone in Q for all $Q \geqslant 0$, not just $Q \geqslant 1$.

Theorem 3.11. For $Q \geqslant 0$, the reliability probability is a strictly monotone decreasing function of Q, for fixed p.

Proof. From (3.10) we know that for any finite G

$$\mathrm{Rel}(G; p, Q_1) > \mathrm{Rel}(G; p, Q_2) \Leftrightarrow T(G; 1 + Q_2 q/p, 1/q) > T(G; 1 + Q_1 q/p, 1/q).$$

But $T(G; x, y) = \Sigma t_{ij} x^i y^j$ and the coefficients t_{ij} are known to be non-negative integers not all of which can be zero. □

4. The dual measure

Consider a random cluster model $\mu = \mu(p, Q)$ on E the edge set E of a planar graph G and let G^* be the dual plane graph with edge set also E identified in the natural and obvious way.

G^* has rank function r^* where r^* is defined by

$$r^*(E \backslash A) = |E| - r(E) - |A| + r(A). \tag{4.1}$$

Note, that

$$T(G; x, y) = T(G^*; y, x). \tag{4.2}$$

For the purposes of this section we identify an edge being *open* with being coloured *black*, *closed* with being coloured *white*. This is helpful and not uncommon in percolation theory, see for example Hammersley and Mazzarino (1983).

We now define the *dual measure* $\hat{\mu}$ of $\mu = \mu(p, Q)$ to be the random cluster measure $\hat{\mu}(\hat{p}, \hat{Q})$ where

$$\hat{p} = \frac{qQ}{p + qQ} \qquad \hat{Q} = Q.$$

thus

$$\hat{\mu}(A) \propto \left(\frac{qQ}{p}\right)^{|A|} Q^{-r(A)}.$$

Proposition 4.3. For any plane graph G and random cluster measure μ

$$P_\mu\{\omega(G) = A\} = P_{\hat{\mu}}\{\omega(G^*) = E \backslash A\}.$$

Corollary 4.4. If G, G^* are dual planar graphs, $\hat{\mu}$ on G^* produces white configurations with exactly the same probability distribution as μ produces black configurations on G.

Proof.

$$\begin{aligned} P_{\hat{\mu}}\{\omega(G) = E \backslash A\} &\propto \left(\frac{qQ}{p}\right)^{|E \backslash A|} Q^{-r^*(E \backslash A)} \\ &\propto \left(\frac{p}{q}\right)^{|A|} Q^{-r(A)} \end{aligned}$$

by substituting from (4.1). □

5. The percolation probability

We now turn to consider percolation in the random cluster model on the square lattice. We adopt the terminology of ordinary ($Q = 1$) percolation as much as possible and in particular follow the notation of Grimmett (1989).

Let Λ_n denote the box on the square lattice having corners $(\pm n, \pm n)$. Let p, Q be fixed and let $\mu_m = \mu_m(p, Q)$ be the sequence of random cluster measures induced by Λ_m, as m runs through the positive integers.

The events in which we have a particular interest are of type $\{0 \rightsquigarrow \partial_n\}$ denoting the event that there is an open path from 0 to ∂_n, the boundary of the box Λ_n.

The results of this section are certainly known, they go back to Fortuin (1972). See Preston (1974) for applications of Holley's theorem and correlation inequalities.

For $Q \geqslant 1$ and $m \geqslant n$

$$\mu_{m+1}\{0 \rightsquigarrow \partial_n\} \geqslant \mu_m\{0 \rightsquigarrow \partial_n\}. \tag{5.1}$$

This is just a special case of the following:

Proposition 5.2. Let G be a finite graph and let H be a subgraph of G on the same vertex set. If μ_G and μ_H denote the random cluster measures induced by G, H respectively for any fixed p and $Q \geqslant 1$, then for any subset $A \subseteq E(H)$

$$\lambda_H(A) \leqslant \lambda_G(A).$$

This is a special case of a more general result:

Proposition 5.3. For any monotone non-decreasing f on the edge set of G, if the value of f is determined by the state of the edges of H, then under the same conditions as above

$$\langle f \rangle_{\mu_H} \leqslant \langle f \rangle_{\mu_G}.$$

Since the quantities in (5.1) are probabilities and thus bounded, we can therefore define

$$\theta_n(p, Q) = \lim_{m\to\infty} \mu_m\{0 \rightsquigarrow \partial_n\}.$$

Now for $m > n$, it is trivial to see that

$$\mu_m\{0 \rightsquigarrow \partial_n\} \leqslant \mu_m\{0 \rightsquigarrow \partial_{n-1}\}.$$

Consequently

$$\theta_n(p, Q) \leqslant \theta_{n-1}(p, Q)$$

and we define

$$\theta(p, Q) = \lim_{n\to\infty} \theta_n(p, Q)$$

to be the *percolation probability* of the model.

Accordingly, for $Q \geqslant 1$, we can define the *critical probability* $p_H(Q)$ by

$$p_H(Q) = \inf\{p : \theta(p, Q) > 0\}.$$

We know therefore that:

(5.4) For $Q \geqslant 1$, $p_H(Q)$ is monotone non-decreasing in Q.

Combining this with the explicit value for $Q = 1$, obtained by Kesten (1980), we have the known result, that

$$p_H(Q) \geqslant \tfrac{1}{2} \qquad \text{for } Q \geqslant 1. \tag{5.5}$$

Following on from the techniques of ordinary percolation we now consider the expected size of a cluster.

For $Q \geqslant 1$, if $\chi(p, Q)$ denotes the expected size of the cluster C through the origin in the random cluster model $\mu(p, Q)$, then

$$\chi(p, Q) = \langle C \rangle = \sum_{y \in Z^2} \mu\{0 \rightsquigarrow y\}$$

where μ is the limit measure, which is well defined for $Q \geqslant 1$, by an argument analogous to the definition of θ (see Aizenman and Grimmett (1991)).

We then define

$$p_T(Q) = \inf\{p : \chi(p, Q) = \infty\}$$

and note immediately that, as with the case $Q = 1$, we have the trivial relationship

$$p_T(Q) \leqslant p_H(Q) \qquad \text{for } Q \geqslant 1. \tag{5.6}$$

I believe that equality holds in (5.6) and give some evidence in support of this in what follows.

Note. In order to clarify the relationship of the above with that of Aizenman *et al* (1988) and Aizenman and Grimmett (1991), it should be pointed out that we are working in the domain of free, not wired, boundary conditions.

6. Sponge crossing probabilities

The sponge percolation model introduced and developed in Seymour and Welsh (1978) has a certain interest in its own right but more importantly it turned out to be a very useful tool in the proofs of various identities about critical probabilities in ordinary bond percolation, see for example Kesten (1980) and Wierman (1981). A very clear treatment is given by Smythe and Wierman (1978) and it is their notation which we follow.

The $m \times n$ *sponge* $T(m, n)$ consists of all vertices and bonds of the square lattice which are contained in the region where $1 \leqslant x \leqslant n$ and $1 \leqslant y \leqslant m$. Each of the m vertices $(1, y)$, for $1 \leqslant y \leqslant m$, on the left side of the sponge is considered to be a source site for fluid which then flows along open bonds in the sponge.

Construct a new graph $G(m, n)$ from the $m \times n$ sponge $T(m, n)$ as follows. Identify the vertices $(1, y)$, $1 \leqslant y \leqslant m$ in a single new vertex x_1. Remove all edges which become loops. Similarly identify all vertices (n, y), $1 \leqslant y \leqslant m$, in a new vertex x_2. Add a special new edge e joining x_1 and x_2. The graph $G(m, n)$ is planar, and its planar dual graph G^* is isomorphic to $G(n-1, m+1)$.

Now consider any assignment w of black and white colours to the edges of $T(m, n)$. There is a path consisting only of white edges from one of the vertices $(1, y)$, $1 \leqslant y \leqslant m$, to one of (n, y), $1 \leqslant y \leqslant m$, if and only if there is a cycle in $G(m, n)$ consisting of the special edge e and otherwise white edges. But by the max-flow min-cut theorem, either there is such a cycle in $G(m, n)$, or there is a cycle in G^* consisting of e and otherwise edges which are black in w, and not both. But since G^* is isomorphic to $G(n-1, m+1)$, and black configurations occur according to the dual probability measure $\hat{\mu}$, what we have shown is the following.

Let $S_{m,n}(p, Q)$ denote the probability that in $G(m, n)$ there is a black path joining the special vertices x_1, x_2 under the measure $\mu_{m,n}$ induced by p, Q and $G(m, n)$. Then:

Theorem 6.1. For positive integers, $m \geqslant 1$, $n \geqslant 2$, and all $p, Q \geqslant 0$

$$S_{m,n}(p, Q) + S_{n-1,m+1}\left(\frac{qQ}{p+qQ}, Q\right) = 1.$$

Suppose now that we define

$$g_n(p, Q) = S_{n,n+1}(p, Q).$$

Then we have the following consequence of theorem 6.1.

Corollary 6.2. For any n, and p, $0 \leqslant p \leqslant 1$, $Q \geqslant 0$

$$g_n(p, Q) + g_n\left(\frac{qQ}{p+qQ}, Q\right) = 1. \tag{6.3}$$

A further consequence of this is that

Corollary 6.4. For any positive integer n and $Q \geqslant 0$

$$g_n\left(\frac{\sqrt{Q}}{1+\sqrt{Q}}, Q\right) = \frac{1}{2}. \tag{6.5}$$

Proof. Follows from (6.3) by solving the equation

$$p = \frac{qQ}{p+qQ}$$

for p. □

Note, also, that if we are just interested in crossing the sponge $T(m,n)$ rather than the graph $G(m,n)$, we have the following consequences of proposition 5.2. Let $\{\rightsquigarrow T(m,n)\}$ denote the event that there is a black path across the $m \times n$ sponge and for any supergraph H of $T(m,n)$ let $\mu_H\{\rightsquigarrow T(m,n)\}$ be its probability under the random cluster measure induced by H with p, Q fixed.

Let $\mu = \mu(p, Q)$ be the limit measure and

$$S_n(p, Q) = \mu\{\rightsquigarrow T(n, n-1)\}.$$

Lemma 6.6. For $Q \geqslant 1$, $S_n(\sqrt{Q}/(1+\sqrt{Q}), Q) \geqslant \frac{1}{2}$.

Proof. Let $\mu = \mu\ (\sqrt{Q}/(1+\sqrt{Q}), Q)$ and note that trivially

$$\begin{aligned}\mu\{\rightsquigarrow T(n, n-1)\} &\geqslant \mu\{x_1 \rightsquigarrow x_2 \text{ in } G(n+1), n)\}\\ &\geqslant \mu_{G(n+1,n)}\{x_1 \rightsquigarrow x_2 \text{ in } G(n+1, n)\}\end{aligned}$$

this last inequality is from proposition 5.2. And this last probability is $g_n(\sqrt{Q}/(1+\sqrt{Q}), Q) = \frac{1}{2}$. □

7. Inequalities for the critical probabilities of the square lattice

In ordinary ($Q = 1$) percolation on the square lattice the tour de force of 20 years effort was the result of Kesten (1980) that the critical probability

$$p_H = p_T = \tfrac{1}{2}. \tag{7.1}$$

I believe that the following Q-extension of this result is true.

Conjecture 7.2. For $Q \geqslant 1$, the critical probabilities $p_T(Q)$ and $p_H(Q)$ are equal and have a common value $\sqrt{Q}/(1+\sqrt{Q})$.

The conjecture is certainly true when $Q = 1$ by virtue of Kesten's theorem that the critical probability of the square lattice is $\frac{1}{2}$. When $Q = 2$, using the relation $p = 1 - \exp(-J)$, it corresponds to a critical value of $\sinh^{-1} 1 = 0.881\,37$ for the critical exponent J, agreeing with the Onsager solution to the Ising model (see Hammersley and Mazzarino (1983, p 209)).

For integer $Q \geqslant 3$ the critical value of $p_c(Q)$ given by the conjecture agrees with the critical points of the Potts model located by singularity based arguments, see for example Baxter (1982) or Hintermann *et al* (1978). However it does not appear easy to make these arguments rigorous in this context, and the situation seems not dissimilar from that in ordinary percolation when it took 16 years before Kesten (1980) and Wierman (1981) were able to give rigorous justifications of the exact values obtained by Sykes and Essam (1964).

Theorem 7.3. For $Q \geqslant 1$, the critical probabilities $p_T(Q)$ and $p_H(Q)$ satisfy

$$p_T(Q) \leqslant \frac{\sqrt{Q}}{1+\sqrt{Q}} \leqslant p_H(Q).$$

Conjecture (7.2) is that each of the above inequalities is an equality. I also believe that it is going to be hard to prove (see the remarks at the end of the paper) but as a partial step towards a proof I obtain the following Q-extension of the relation $p_T + p_H \leqslant 1$ of ordinary square lattice percolation.

Consider the family of functions $f_Q : [0, 1] \to [0, 1]$ defined for $Q \geqslant 0$ by

$$f_Q(x) = \frac{Q(1-x)}{x + Q(1-x)}.$$

For $Q > 0$, f_Q is monotone decreasing, and convex ($Q > 1$) or concave ($Q < 1$).

Theorem 7.4. For $Q \geqslant 1$, $p_H(Q)$ and $p_T(Q)$ are related by

$$p_H(Q) \leqslant f_Q(p_T(Q)).$$

Proof of Theorem 7.3. First define $p_S(Q)$, yet a third critical probability, by

$$p_S(Q) = \lim \inf\{p \geqslant 0 : S_n(p, Q) > 0\}.$$

Then we know from (6.6) that

$$p_S(Q) \leqslant \sqrt{Q}/(1 + \sqrt{Q}). \tag{7.5}$$

Our result will follow from showing that

$$p_T(Q) \leqslant p_S(Q). \tag{7.6}$$

To prove this let B_n denote the $n \times (n-1)$ sponge and let

$$R = \{(x, y) : x = n - 1,\ 1 \leqslant y \leqslant n\}$$

be the set of vertices on the right border of B_n. For each vertex $(1, i)$, $i \leqslant n$, let $C_i(w)$ denote the cluster containing $(1, i)$. If $p < p_T$ then the expected cluster size is finite.

Hence, we have

$$\begin{aligned} P_\mu\{(1, i) \overset{B_n}{\rightsquigarrow} R\} &\leqslant P_\mu\{|C_i| \geqslant n - 1\} \\ &\leqslant \langle |C_i| \rangle_\mu/(n-1). \end{aligned}$$

Also, since $Q \geqslant 1$, we can use proposition 5.2 and write

$$\begin{aligned} S_n(p, Q) &\leqslant P_\mu\{\bigcup_{i=1}^{n} (1, i) \overset{B_n}{\rightsquigarrow} R\} \\ &\leqslant \sum_{i=1}^{n} P_\mu\{(1, i) \overset{B_n}{\rightsquigarrow} R\} \\ &\leqslant \sum_{i=1}^{n} P_\mu\{|C_i| \geqslant n - 1\} \\ &= \sum_{i=1}^{n} \sum_{j=n-1}^{\infty} P\{|C_i| = j\} \\ &\leqslant \sum_{j=n-1}^{\infty} j P\{|C_i| = j\}. \end{aligned}$$

But $\langle C_i \rangle < \infty$ implies that this last sum tends to zero as $n \to \infty$ and thus $p_T(Q) \leqslant p_S(Q)$ as required.

It remains to prove that for $Q \geqslant 1$

$$p_H(Q) \geqslant \frac{\sqrt{Q}}{1+\sqrt{Q}}. \tag{7.7}$$

This is the Q-analogue of one of the first exact results in percolation theory namely Harris' theorem (Harris 1960) that in bond percolation on the square lattice the critical probability is at least $\frac{1}{2}$. I cannot see how to modify Harris' proof to obtain (7.7), nor can I base a proof on the proof of the same result by Smythe and Wierman (1978) since this depends on lemmas from Seymour and Welsh (1978) which do not appear easy to generalize for $Q \neq 1$. Instead I sketch a completely different proof. The argument is as follows.

First we note that since $Q \geqslant 1$, the main result of Gandolfi *et al* (1992) gives:

(7.8) With probability one, if there is an infinite cluster it is unique.

Now let $p_0 = \sqrt{Q}/(1+\sqrt{Q})$ and suppose that $\theta(p_0, Q) > 0$. Then the argument of Zhang as given in Grimmett (1989, p 195) goes through, almost word for word, replacing the critical Bernoulli measure ($p = \frac{1}{2}$, $Q = 1$) with $\mu(\sqrt{Q}/(1+\sqrt{Q}), Q)$ and we get the contradiction that there almost surely exist two infinite (closed = white) clusters on the dual lattice. Hence the initial assumption $\theta(p_0, Q) > 0$ is incorrect. □

Proof of Theorem 7.4. Fix $Q > 1$, let $\epsilon > 0$, let $p_T = p_T(Q)$ and $p = p_T - \epsilon$, so that $f(p) > f(p_T)$.

If R denotes the right border of the sponge $T(2n, n)$ then since $Q > 1$, we can use proposition 5.2 to get

$$\begin{aligned} P_\mu\{\rightsquigarrow T(2n,n)\} &\leqslant P_\mu\{\bigcup_{i=1}^{2n}\{(1,i) \rightsquigarrow R\}\} \\ &\leqslant \sum_{i=1}^{2n} P_\mu\{(1,i) \rightsquigarrow R\} \\ &\leqslant 2n P_\mu\{|C_0| \geqslant n\} \end{aligned}$$

where C_0 denotes the cluster through the origin and $\mu = \mu(p, Q)$ is the limiting measure. The argument given by Smythe and Wierman (1978, pp 33–4) now goes through almost word for word with P_μ replacing the standard (Bernoulli) probability they use, to obtain the result that in the dual lattice (with in this case the dual measure $\hat{\mu}$) there is strictly positive probability that the origin of the dual lattice belongs to an infinite cluster.

But the dual measure $\hat{\mu} = \mu(f(p), Q)$ and thus $p_H(Q) \leqslant f(p)$. But since $p = p_T - \epsilon$ where ϵ is arbitrary and f is continuous, the result follows. □

I conclude with a brief discussion of the difficulties involved in trying to extend Kesten's theorem (for $Q = 1$) to general $Q \geqslant 1$.

Although, when $Q \geqslant 1$, the resulting μ is an FKG measure and this is used extensively, all proofs that I know depend in some part on being able to say that events depending on disjoint sets of arcs are independent—positive correlation is not enough. In particular, annulus arguments and the second Borel–Cantelli theorem (or something like it) seem essential.

Nevertheless, large chunks of the existing proofs do go through relatively unchanged and I see some hope.

For the case $0 < Q < 1$, the definitions of θ and the corresponding critical probabilities have not even been given in this paper. They are harder to formulate and the FKG property, a basic tool in the preceding work fails in this domain.

Note also that similar methods can be applied and inequalities obtained for the other planar lattices and for mixed (anisotropic) percolation in which bonds in different directions have different probabilities. For example suppose that the horizontal and vertical bonds of the square lattice have probabilities p_1, p_2 respectively so that μ is given by

$$\mu(A) \propto \left(\frac{p_1}{q_1}\right)^{|A_H|}\left(\frac{p_2}{q_2}\right)^{|A_V|} Q^{-k(A)}$$

where A_H, (A_V) are the sets of horizontal (vertical) edges of A, and $q_i = 1 - p_i$. Then similar arguments would suggest a critical surface (as defined in Grimmett (1989)) given by

$$(Q-1)p_1p_2 - Q(p_1+p_2) + Q = 0. \tag{7.9}$$

Similarly, the above duality, and star-triangle techniques such as were used by Wierman (1981) for the classical percolation model and described in Wu (1982) for the Q-state Potts model, suggest that the critical probabilities $p_c = p_c(Q)$ for the random cluster model on the triangular and hexagonal lattices have to satisfy the following cubic equations.

For the triangular lattice, and for $Q \geqslant 1$, the critical probability should satisfy

$$p^3(Q-2) - 3p^2(Q-1) + 3pQ = Q. \tag{7.10}$$

For the hexagonal lattice, and for $Q \geqslant 1$, the critical probability should satisfy

$$p^3(Q^2-3Q+1) - 3p^2(Q^2-2Q) + 3pQ(Q-1) = Q^2. \tag{7.11}$$

However, proving any of (7.9)–(7.10) rigorously will be even more difficult.

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